THE PSEUDOTURBULENT DIFFUSION OF PARTICLES IN HOMOGENEOUS SUSPENSIONS

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The effective diffusion coefficients for suspended particles caused by their pseudoturbulent pulsations, are treated. Derivatives of the dynamic variables which determine the average motion of the locally homogeneous suspension are neglected.

By definition [1] the tensor of particle diffusion coefficients, for the case in which the diffusion is brought about by the random motion of particles, can be represented in the form

$$D_{ij} = \frac{1}{2}\int_0^\infty (R_{wi,\ wj}(\tau) + R_{wj,\ wi}(\tau))\, d\tau \tag{1}$$

where the integrand consists of components of the tensor of Lagrangian correlation functions for the particle velocity **w'**. These quantities can be written in the form

$$R_{wi,\ wj}(\tau) = \int e^{i\omega\tau} \Psi_{wi,\ wj}(\omega, \mathbf{k})\, d\omega d\mathbf{k}. \tag{2}$$

Here the integration is carried out over all frequencies ω and all wave-space **k**, while $\Psi_{Wi,Wj}(\omega,k)$ is the spectral tensor of the random vector **w'**, introduced in [2]. This tensor is defined in [2] in such a way that its integral with respect to wave-space is the ordinary Lagrangian spectral tensor of particle velocity.

It can easily be seen from [2] that the tensor $\Psi_{Wi,Wj}(\omega, k)$, considered as a function of ω, satisfies all the conditions necessary for changing the order of integration with respect to τ and ω in (1) and (2). Changing the order of integration and using the Fourier integral expansion for the δ- function, we obtain the following relation for the pseudoturbulent particle diffusion tensor from (1) and (2):

$$D_{ij} = \frac{\pi}{4}\int (\Psi_{wi,\ wj}(0, \mathbf{k}) + \Psi_{wj,\ wi}(0, \mathbf{k}))\, d\mathbf{k}. \tag{3}$$

The usual means [1] of expressing the quantities $\Psi_{Wi,Wj}(\omega, \mathbf{k})$ is in terms of average products of corresponding components of the spectral measure dZ_W of the random process **w'**. Equations for dZ_W and spectral measures of other random quantities characterizing pseudoturbulance in a suspension are obtained in [2]. It can easily be seen from (3) that in the present paper we have only to consider these equations for zero frequency ω and only for a steady-state flow without gradient, when the scales of the average motion are much longer than the scale of the pseudoturbulance, i.e., when we can neglect derivatives of dynamic variables characterizing suspension flow in the continuous approximation. It was shown in [2] that this latter corresponds to the familiar Euler approximation in the hydrodynamic approximation of a single-phase fluid. For $\omega = 0$ we have the following equations from [2]:

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$$\mathbf{uk}\,dZ_\rho = (1-\rho)\,\mathbf{k}dZ_v$$
$$\varkappa(\beta K+\zeta)\,d\mathbf{Z}_w - \varkappa\beta K d\mathbf{Z}_v = -\frac{i}{d_2}\,\mathbf{k}dZ_p + \varkappa\beta\frac{dK}{d\rho}\,\mathbf{u}dZ_\rho$$
$$\varkappa\rho\zeta d\mathbf{Z}_w + \varkappa\,[i(1-\rho)\,\mathbf{uk} + \varkappa\nu_0 Sk^2]\,d\mathbf{Z}_v = -\frac{i}{d_2}\,\mathbf{k}dZ_p - \frac{\varkappa}{3}\nu_0 S(\mathbf{k}dZ_v)\,\mathbf{k} \qquad (4)$$
$$\beta = \frac{9\nu_0}{2a^2} \qquad \zeta = \frac{c}{4}\frac{N}{n}\left[\xi + 4\gamma\eta\left(\frac{n}{N}\right)^{1/3}\right], \quad \gamma = \frac{9}{2a}\left(\frac{\nu_0}{\pi}\right)^{1/2}, \quad \varkappa = \frac{d_2}{d_1}.$$

Here the same symbols are adopted as in [2], but the average sign $\langle\rangle$ is omitted from the symbols for dynamic variables to simplify the notation. In the derivation of (4) an expression was used for the interaction force between phases, valid for $R \ll 1$, where $R = 2a\omega/\nu_0$ is the Reynolds' number. This expression characterizes the relative flow of the liquid phase around individual particles.

It is convenient to pass to dimensionless variables, introducing a characteristic velocity u, characteristic length a, and consequently a characteristic time a/u. In what follows, the treatment is presented for dimensionless variables, which are the ratios of the corresponding variables to dimensional quantities constructed from the characteristic scales selected. Thus, for example, the dimensional velocities and diffusion coefficients are divided by u and ua respectively, the dimensional frequency and wave-vector by u/a and $1/a$, etc. The only exception is the dimensional spectral measure of pressure perturbations in the flow dZp, which is divided by $d_1\beta u a K$. When quantities are rendered dimensionless in this way, the form of Eqs. (1)-(3) is retained, and instead of (4) we have

$$\mathbf{k}dZ_v = \frac{\mathbf{u}_0\mathbf{k}}{1-\rho}dZ_\rho, \quad (1+\alpha)\,d\mathbf{Z}_w - d\mathbf{Z}_v = -i\mathbf{k}dZ_p + \frac{d\ln K}{d\rho}\,\mathbf{u}_0 dZ_\rho$$
$$\rho\varkappa d\mathbf{Z}_w + [i(1-\rho)\,r\mathbf{u}_0\mathbf{k} + sk^2]\,d\mathbf{Z}_v = -i\mathbf{k}dZ_p - {}^1/_3\,s(\mathbf{k}dZ_v)\,\mathbf{k} \qquad (5)$$
$$\mathbf{u}_0 = \frac{\mathbf{u}}{u}, \quad s = \frac{2}{9}\frac{S}{K}, \quad r = \frac{R}{9K} = \frac{2}{9K}\frac{ua}{\nu_0}, \quad \alpha = \frac{\zeta}{\beta K}.$$

The parameter α in (5) characterizes the ratio of dissipative forces arising from the instantaneous acceleration of the associated fluid mass due to instantaneous velocity changes of colliding particles, to the viscous interaction forces between phases [2]. An order-of-magnitude estimate was obtained in [2] for α on the basis of a model in which there are elastic collisions in a gas of particles having an isotropic Maxwell velocity distribution. In dimensionless form this gives us

$$\alpha = \frac{1}{3K}\left(\frac{3}{\pi}\langle w'^2\rangle\right)^{1/4}(\rho\chi R)^{1/2}\left[\frac{\xi}{3}\left(\frac{3}{\pi}\langle w'^2\rangle\right)^{1/4}(\rho\chi R)^{1/2} + 3\eta\left(\frac{2}{\pi}\right)^{1/2}\right]. \qquad (6)$$

The symbols of paper [2] are also retained here.

From physical considerations it is natural to expect that "collisional" dissipation in a dispersive system is relatively small, i.e., $\alpha \ll 1$, at least for systems in which the concentration is not very close to the concentration of a granular layer in the close-packed state. This conclusion results from the following considerations in particular.

1. By its nature collisional dissipation is proportional to the collision frequency in the suspension of and the size of velocity discontinuity for colliding particles, i.e., it is always small for rarefied suspensions.

2. The collisions of particles suspended in the fluid usually lead to quite smooth, rather than abrupt, changes of particle velocity. This is associated with the considerable pressure increase in the fluid layer between particles as they approach each other, and the necessity for "squeezing out" this layer before there can be direct contact of the particles. A similar effect also occurs when a particle approaches a solid wall [3], and in lubrication processes, when the part of the fluid layer is played by the lubricating fluid in the space between the journal and bearing [4]. We can thus assume that the estimate (6) based on a model of purely elastic collisions between particles, is higher by an order of magnitude even for suspensions which are not very concentrated.

3. Finally we can expect a substantial effect from direct particle collisions (contacts) predominantly in suspensions of coarse and heavy particles in fluids of low density and viscosity, particularly in gases.

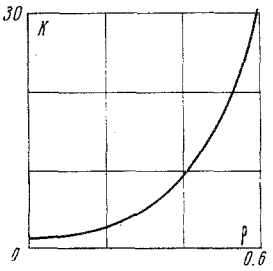

Fig. 1

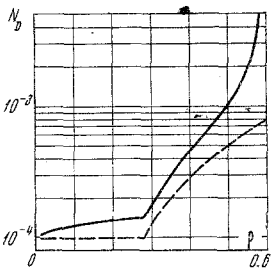

Fig. 2

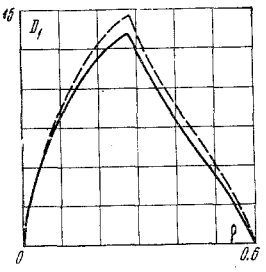

Fig. 3

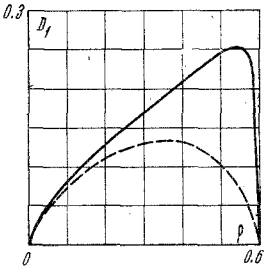

Fig. 4

However, such systems are usually locally inhomogeneous, the behavior of neighboring particles is strongly correlated, and the number of effective collisions should be considerably lower than in locally homogeneous dispersive systems, to which estimate (6) refers. This also leads to a considerable decrease in collisional dissipation.

We note that for small δ and R the smallness of α is also confirmed by estimate (6). We refrain from considering suspensions in states close to the close-packed state, when collisional dissipation can, generally speaking, be considerable, and assume that $\alpha \approx 0$.

Solving Eqs. (5) for $\alpha = 0$, we have

$$d\mathbf{Z}_w|_{\omega=0} = \left\{\frac{d\ln K}{d\rho}\mathbf{u}_0 + \left[i(1-\rho)r\mathbf{u}_0\mathbf{k} + 1 + \frac{4}{3}sk^2\right]\frac{r_0\mathbf{k}}{1-\rho}\frac{\mathbf{k}}{k}\right\}dZ_\rho . \tag{7}$$

It is convenient to carry out the analysis using the principal axes of the tensor D as coordinates, and so we let the coordinate axis x_1 lie along the vector $\mathbf{u}'_0$. We then obtain the following expressions for the diagonal components of $\Psi_{W,W}(0,\mathbf{k})$ from (7):

$$\begin{aligned}\Psi_{w1,\,w1}(0,\mathbf{k}) &= \left[\left(\frac{d\ln K}{d\rho} + \frac{1+{}^4/_3 sk^2}{1-\rho}\frac{k_1^2}{k^2}\right)^2 + r^2k_1^2\frac{k_1^2}{k^4}\right]\Psi_{\rho,\rho}(0,\mathbf{k})\\ \Psi_{wj,\,wj}(0,\mathbf{k}) &= \left[\left(\frac{1+{}^4/_3 sk^2}{1-\rho}\right)^2 + r^2k_1^2\right]\frac{k_1^2k_j^2}{k^4}\Psi_{\rho,\rho}(0,\mathbf{k}) .\end{aligned} \tag{8}$$

The summation with respect to j in (8) is not carried out; the cross-components of the spectral density tensor are not written down, since they give zero when integrated over wave space in (3). The spectral density $\Psi_{p,\rho}(\omega,\mathbf{k})$ for perturbations of the concentration ρ' is defined in dimensionless frequency space ω and wave vector space k; we use an expression for it which follows from results [2]

$$\begin{aligned}&\Psi_{\rho,\rho}(0,\mathbf{k}) = \frac{(\mathbf{Dkk})^{-1}}{\pi}\Phi_{\rho,\rho}(\mathbf{k}), \quad \Phi_{\rho,\rho}(\mathbf{k}) = \frac{\Phi}{k_0^3}Y(k_0-k)\\ &\Phi = \frac{3}{4\pi}\rho^2\left(1-\frac{\rho}{\rho_*}\right), \quad k_0 = \left(\frac{3\pi\rho}{2}\right)^{1/3}\left(1-\frac{\rho}{\rho_*}\right)^{-1/3}, \quad Y(x) = \begin{cases}1, & x>0\\ 0, & x<0.\end{cases}\end{aligned} \tag{9}$$

Of course the expression for k_0 in (9) refers to locally homogeneous suspensions only. For locally inhomogeneous dispersive systems k_0 must be treated as some free parameter, which coincides in order of magnitude with the ratio of the particle radius a to the mean radius of inhomogeneities arising in the flow [5].

It is clear from (3), (8) and (9) that the pseudoturbulent diffusion of particles is nonisotropic. It is axisymmetric, and the preferred direction is the direction of interphase slipping $\mathbf{u}_0$. On integrating we obtain the following equations for the effective dimensionless coefficients of pseudoturbulent particle diffusion in the longitudinal and transverse directions:

$$D_1D_2 = \frac{2\pi\Phi}{k_0^2}\gamma^2\left[\left(\frac{d\ln K}{d\rho}\right)^2 J_0 + \frac{2}{1-\rho}\frac{d\ln K}{d\rho}\left(1+\frac{4}{9}sk_0^2\right)J_2 + \frac{1}{(1-\rho)^2}\left(1+\frac{8}{9}sk_0^2+\frac{16}{45}s^2k_0^4\right)J_4 + \frac{r^2}{3}k_0^2J_6\right]$$

$$D_2^2 = \frac{\pi\Phi}{k_0^2}\gamma^2\left[\frac{1}{(1-\rho)^2}\left(1+\frac{8}{9}sk_0^2+\frac{16}{45}s^2k_0^4\right)(J_2-J_4) + \frac{r^2}{3}k_0^2(J_6-J_4)\right] \qquad (10)$$

$$\gamma^2 = \frac{D_2}{D_1-D_2}\left(\frac{D_1}{D_2} = \frac{1+\gamma^2}{\gamma^2}\right), \quad J_n = \int_0^1 \frac{t^n dt}{t^2+\gamma^2}.$$

In view of the approximations made in deriving (4) and (5), expressions (10) may be taken to be valid for $R \lesssim 1$, which corresponds normally to locally homogeneous suspensions. In this case it is permissible firstly to use the expression for k_0 from (9), and secondly to neglect the term involving r^2 in (10). It is not difficult to see that neglect of this term is equivalent to neglecting the momentum of the liquid phase, as is done, for example, in [5]. Thus this approximation turns out to be valid for fairly small values of the Reynolds' number R.

Equations (10) can easily be separated. In fact, from (10), we have the following transcendental equation for the quantity γ:

$$\frac{2\gamma^2}{1+\gamma^2} = \frac{(1+{}^8/_9\,sk_0^2+{}^{16}/_{45}\,s^2k_0^4)(J_2-J_4)}{n^2J_0+2n(1+{}^4/_9\,sk_0^2)J_2+(1+{}^8/_9\,sk_0^2+{}^{16}/_{45}\,s^2k_0^4)J_4}. \qquad (11)$$

Here the function

$$n(\rho) = (1-\rho)\frac{d\ln K}{d\rho} \qquad (12)$$

has been introduced, and has a particularly simple form for approximations like $K(\rho) \approx (1-\rho)^{-n}$.

Neglecting terms involving r^2, we have, for example, the following expression for the longitudinal diffusion coefficient from (10):

$$D_1 = \frac{(1+\gamma^2)(\pi\Phi)^{1/2}}{\gamma k_0(1-\rho)}\left(1+\frac{8}{9}sk_0^2+\frac{16}{45}s^2k_0^4\right)^{1/2}(J_2-J_4)^{1/2}. \qquad (13)$$

The symbol γ, which appears, in particular, in the integrals J_n, is taken to mean the single root of Eq. (11).

Analysis shows that the quantity γ is usually very small. An approximate expression for γ can then be easily obtained from (11). This parameter ceases to be small only for concentrations which are very close to the concentration of a close-packed layer of particles ρ_*, since for $\rho \to \rho_*$ the quantity s in (11) and (13) ceases to be finite, and k_0 from (9) tends to infinity. It can be assumed that for such values of ρ, the theory developed above ceases to be valid in general since collisional dissipation was neglected.

Neglect of viscous stresses in a dissipative medium would mean s = 0 in expressions (11)-(13). It can easily be seen that this approximation is quite justified for small and moderate values of ρ, but ceases to be valid for $\rho \to \rho_*$.

We must have the functions $K(\rho)$ and $S(\rho)$ for concrete calculations. The function $K(\rho)$ describes the increase in effective hydraulic resistance of a particle because of the restriction of particle flow in a fixed nozzle, where the motions are not pseudoturbulent. The effective resistance of a layer of randomly moving particles, a pseudoliquefied layer, for which there is a great deal of experimental material [6], can be quite different from the resistance in a motionless nozzle of the equivalent porosity [5, 6]. There is much less experimental data on the quantity $K(\rho)$ in a motionless layer and, what is most important, all this data was obtained for comparatively high values of ρ, as a rule close to ρ_*. One of the most widely used func-

tions $K(\rho)$ for a layer of hard particles was obtained by Ergun [7], who generalized a large amount of the available experimental data. For small Reynolds' numbers R we have from [7]

$$K(\rho) \approx \frac{25}{3}\frac{\rho}{(1-\rho)^2}.$$

This function is valid only for $\rho > \rho_0 \approx 0.25$-$0.30$; for $\rho \to 0$ it does not give the required limit $K(\rho) \to 1$. In order to obtain a relation approximately valid over the whole interval from 0 to ρ_*, we must supplement this function in the region $\rho < \rho_0$ by the function

$$K(\rho) \approx (1-\rho)^{-n}$$

choosing the constants ρ_0 and n so that $K(\rho)$ and its first derivative are continuous at the point $\rho = \rho_0$. This leads to the equations

$$\rho_0 = \frac{1}{n-1}, \qquad \frac{25}{3(n-1)} = \left(\frac{n-1}{n-2}\right)^{n-2}$$

which can be solved to give $n \approx 4.58$, $\rho_0 \approx 0.28$.

In what follows we shall use the approximation

$$K(\rho) = \begin{cases} (1-\rho)^{-4.58}, & \rho < 0.28 \\ {}^{25}/_{8}\,\rho\,(1-\rho)^{-2}, & \rho > 0.28 \end{cases}. \tag{14}$$

This function is shown in Fig. 1; a value $\rho_* = 0.60$ was assumed in the calculations.

The function $S(\rho)$ describes the increase in fluid viscosity as the result of distortion of the steamlines in the presence of a system of suspended particles. To be specific we use the approximation

$$S(\rho) = (1-\rho)^{-2.5} \tag{15}$$

which passes to the familiar Einstein limit $S \approx 1 + 2.5\,\rho$ when $\rho \to 0$.

The ratio $N_D = \gamma^2(1+\gamma^2)^{-1}$ of the transverse and longitudinal coefficients for pseudoturbulent particle diffusion, calculated from (11) using (14) and (15), is shown by the continuous line in Fig. 2 for $\rho_* = 0.60$. The dashed line shows N_D as a function of ρ, corresponding to the "nonviscous" model with $s = 0$. Pseudoturbulent diffusion of solid particles is, clearly, strongly anisotropic; longitudinal diffusion turns out to be 10^3-10^4 times stronger than transverse diffusion. In many applications it is sensible to talk about the longitudinal diffusion of particles only.

The functions D_1 from (13) are given in Fig. 3 for $s \neq 0$ (continuous curve) and $s = 0$ (dashed line). The quantity D_2 can also be found easily from the data of Figs. 2 and 3. The knees in the N_D and D_1 curves appear as the result of the discontinuity in the second derivative of $K(\rho)$ for $\rho = 0.28$.

A system of gas bubbles suspended in a fluid is of particular interest. Naturally we assume that in the process of pseudoturbulent motion the bubbles do not lose their individuality, i.e., they do not coalesce, do not break up, and do not stick together to form foam. This requires, in particular, that the surface tension at the fluid-gas boundary should be quite large and the bubbles themselves small.

In the case we can take $K(\rho) \approx 1 = \text{const}$; this result comes, for example, from calculations of the hydraulic resistance of a lattice of bubbles based on the mesh model. We use the approximation $S(\rho) = (1-\rho)^{-1}$ for $S(\rho)$, so that we have the familiar result of Gut and Mark, $S \approx 1 + \rho$ when $\rho \to 0$. Calculations of γ from Eq. (11) give $\gamma = 0.8546$, $N_D = 0.4221$. Pseudoturbulent diffusion of bubbles is considerably less anisotropic than for a suspension of solid particles. This is so chiefly because in a suspension of bubbles, there is no force arising from the nonlinearity of K as a function of ρ, and from fluctuations of ρ' in the system. In this case the ratio N_D turns out to be universal. It is independent of ρ and of the physical parameters of the phases of the emulsion. The corresponding value of N_D for a suspension of solid particles is also independent of the physical parameters, but is a function of ρ.

The coefficient D_1 for gas bubbles is given in Fig. 4 as a function of ρ for $s \neq 0$ and $s = 0$ by the continuous and dashed curves respectively.

When the complicated nature of the systems under investigation and the ambiguity of experimental data on diffusion in these systems is taken into account (see, for example, [6]), the small quantitative difference between models with $s \neq 0$ and $s = 0$ can in many cases be neglected.

We note that for $s = 0$ the results (11) and (12) do not in general agree with the corresponding results for the pseudoturbulent diffusion of particles in a gas obtained in [5] on the basis of a model in which the momentum and viscous stresses in the gas were neglected a priori [5]. This is connected with the fact that

$$\lim_{\omega \to 0,\ \varepsilon \to 0} \Psi_{w,\,w}(\omega,\ k) \neq \lim_{\varepsilon \to 0,\ \omega \to 0} \Psi_{w,\,w}(\omega,\ k)$$

where ε denotes the set of parameters characterizing the effect of momentum and viscous stresses in the gas. Thus the model in [5], although leading in a series of cases to qualitatively correct results, is somewhat formal in context.

Experiments for determining effective particle diffusion coefficients have been carried out basically for pseudoliquefied systems. In view of the diversity of devices and layers used, and methods for measuring or calculating (from the measured viscosity or thermal diffusivity of the layer) the coefficients of longitudinal and transverse diffusion, the results obtained by different authors exhibit fairly considerable discrepancies [6]. There are, moreover, in all such systems, circulation currents of both phases, caused by friction in the neighborhood of the walls. The component u_n of the relative viscosity of the liquid phase of this circulatory flow, normal to the direction of the basic flow, is lower than u by roughly an order of magnitude. Exactly the same relation is observed for measured values of the transverse and longitudinal particle-diffusion coefficients in narrow layers [6]. Thus in the experiments one is actually dealing not with the transverse diffusion coefficient in the sense of this paper, but rather with the longitudinal (relative to u_n) particle-diffusion coefficient.

At present one can only talk about experimentally verifying the expression for the longitudinal pseudoturbulent particle-diffusion coefficient, which is much less dependent on the presence of circulation. It is not difficult to establish that the values of this coefficient, calculated from the results of this paper, coincide in order of magnitude with the values determined experimentally (see review in [7]).

The conclusion that longitudinal diffusion has a marked predominance over transverse diffusion also agrees with direct observations of particle pulsations in comparatively rarified systems. It suffices to say that the first papers on this topic were generally concerned only with longitudinal pulsations (see, for example, the description of the experiments of A. K. Bondareva in [6]). However, it should be noted that the physical reason why longitudinal diffusion predominates can be found not only in the velocity anisotropy of pseudoturbulent pulsations, but also in the anisotropy of the corresponding mixing lengths. If the mixing length in the transverse direction is of the order of the mean distance between particles in suspension, then it can be considerably lower in the longitudinal direction [6].

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